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The sum of Rademacher functions and Hausdorff dimension

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Abstract

For $0 < \alpha < 1$, let $f_\alpha(x) = \sum_{i=1}^{\infty} 2^{-\alpha i} R_i(x)$ for $0 \leqslant x < 1$, where $\{R_i\}_{i=1}^{\infty}$ is the sequence of Rademacher functions. We give a class of f_α so that their graphs have Hausdorff dimension $2-\alpha$. The result is closely related to the corresponding unsolved question for the Weierstrass functions.

1. *Introduction*

For $0 < \alpha < 1$, let

$$f_\alpha(x) = \sum_{i=1}^{\infty} 2^{-\alpha i} R_i(x) \quad (0 \leqslant x < 1),$$

where R_i, for $i = 1, 2, \ldots$, denotes the Rademacher functions: $R_1(x)$ is defined on $\mathbb{R}$ with period 1, takes values 1 and -1 on the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ respectively, and $R_i(x) = R_1(2^{i-1}x)$ for $i > 1$. In this note we will give a sufficient condition to determine the Hausdorff dimension of the graph Γ_{f_α} of f_α (denoted by $\dim \Gamma_{f_\alpha}$). This is closely related to the well known open problem whether the graph of the Weierstrass function

$$W_\alpha(x) = \sum_{i=1}^{\infty} \lambda^{-\alpha i} \sin(\lambda^i \pi x) \quad (0 \leqslant x < 1),$$

where $\lambda > 1$, has Hausdorff dimension $2-\alpha$: see [**3**], p. 114 and [**8**]. (Note that $R_i(x) = \operatorname{sgn} \sin(2^i \pi x)$ except when x is a zero of $\sin(2^i \pi x)$.)

Let $F_\alpha(y) = |\{x \in [0, 1) : f_\alpha(x) < y\}|$ be the distribution function of f_α, where $|A|$ denotes the Lebesgue measure of a measurable subset A of $\mathbb{R}$. Among the other results we prove

THEOREM 1·1. *Suppose that F_α is absolutely continuous and $F'_\alpha \in L^p$ for some $p > 1$. Then* $\dim \Gamma_{f_\alpha} = 2-\alpha$.

The function f_α can be considered as the random variable of the sum of a sequence of independent Bernoulli trials. The distribution F_α can hence be obtained as the infinite convolution of the Bernoulli distributions. It follows from a theorem of Jessen and Wintner [**6**] that F_α is either absolutely continuous or purely singular. The determination of which type is however, very difficult (see Erdös [**2**], Garsia [**4**], Kahane and Salem [**7**], Salem [**9**] and Wintner [**10**]). Although it is known that F_α is absolutely continuous for some α and that F_α is purely singular for some α, a complete description for $0 < \alpha < 1$ is still unknown.

Theorem 1·1 is proved in Section 2; the proof actually goes through with a weaker assumption on F_α, namely, that F_α satisfies certain integrated Lipschitz condition (Theorem 2·6). The known results on the absolute continuity of F_α, the relevance of the above integrated Lipschitz condition, and other remarks on the dimension of Γ_{f_α} are discussed in Section 3.

2. *The theorem*

Let $\mathbb{N} = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$ and $0 \leqslant k \leqslant 2^n - 1$, let $I_{n,k}$ be the dyadic intervals $[2^{-n}k, 2^{-n}(k+1))$ in $[0, 1)$, and let $\mathcal{S}$ be the class of dyadic squares in $[0,1) \times \mathbb{R}$. For $s > 0$, we use $\mathcal{H}^s$ to denote the Hausdorff s-dimensional measure on $\mathbb{R}^2$, i.e. for any subset E in $\mathbb{R}^2$,

$$\mathcal{H}^s(E) = \lim_{\delta \to 0_+} \inf\{\sum_i (\operatorname{diam} U_i)^s : E \subseteq \bigcup_i U_i, \operatorname{diam} U_i < \delta\},$$

where $\{U_i\}$ are open subsets in $\mathbb{R}^2$. The Hausdorff dimension of E, denoted by $\dim E$, is defined by

$$\dim E = \inf\{s > 0 : \mathcal{H}^s(E) = 0\}$$

(which also equals $\sup\{s > 0 : \mathcal{H}^s(E) = \infty\}$). It is known that if we replace the U_i by dyadic squares, the dimension of E is unchanged.

In this section we will fix $0 < \alpha < 1$. We can hence use f, F and Γ to denote f_α, F_α and Γ_{f_α} respectively without causing confusion.

LEMMA 2·1. *For $0 < \alpha < 1$, $\mathcal{H}^{2-\alpha}(\Gamma) < \infty$, and hence $\dim \Gamma \leqslant 2 - \alpha$.*

Proof. The proof is the same as in [3], theorem 8·1 for Lipschitz functions.

Let $I_{n,k}$ be a dyadic interval in $[0, 1)$, let f be restricted on $I_{n,k}$ and define the corresponding distribution function $F_{n,k}$ by

$$F_{n,k}(y) = |\{x \in I_{n,k} : f(x) < y\}|.$$

LEMMA 2·2. (i) *For $x \in I_{n,k}$, $f(x) = c_{n,k} + 2^{-\alpha n} f(2^n(\alpha - 2^{-n}k))$ for some constant $c_{n,k}$, and* (ii) *for $y \in \mathbb{R}$, $F_{n,k}(y) = 2^{-n}F(2^{\alpha n}(y - c_{n,k}))$.*

Proof. (i) For $x \in I_{n,k}$, using the periodicity of R_i we can write

$$f(x) = \sum_{i=1}^{n} 2^{-\alpha i} R_i(x) + 2^{-\alpha n} \sum_{i=1}^{\infty} 2^{-\alpha i} R_i(2^n(x - 2^{-n}k)) = c_{n,k} + 2^{-\alpha n} f(2^n(x - 2^{-n}k)).$$

(ii) Let $y \in \mathbb{R}$. Then by (i) we have

$$\begin{aligned} F_{n,k}(y) &= |\{x \in I_{n,k} : c_{n,k} + 2^{-\alpha n} f(2^n(x - 2^{-n}k)) < y\}| \\ &= |\{x \in I_{n,k} : f(2^n(x - 2^{-n}k)) < 2^{\alpha n}(y - c_{n,k})\}| \\ &= |\{2^{-n}(u + k) \in I_{n,k} : f(u) < 2^{\alpha n}(y - c_{n,k})\}| \\ &= 2^{-n}|\{u \in [0,1) : f(u) < 2^{\alpha n}(y - c_{n,k})\}| = 2^{-n}F(2^{\alpha n}(y - c_{n,k})). \end{aligned}$$

In the following we will give a simple proof of a special case of Theorem 1·1, which also serves as motivation for the more elaborate proof in Lemma 2·5.

Let $P: \mathbb{R}^2 \to \mathbb{R}$ be the natural projection defined by $P(x, y) = x$.

THEOREM 2·3. *If F is absolutely continuous and $F' \in L^\infty$, then $0 < \mathscr{H}^{2-\alpha}(\Gamma) < \infty$, and hence* $\dim \Gamma = 2-\alpha$.

Proof. Let M be the essential supremum of F', and let $\{S_i\} \subseteq \mathscr{S}$ be an arbitrary cover of Γ, where $S_i = I_{n_i,k_i} \times [y_i, y_i + 2^{-n_i})$. Then by Lemma 2·2 we have

$$\begin{aligned} |P(S_i \cap \Gamma)| &= F_{n_i,k_i}(y_i + 2^{-n_i}) - F_{n_i,k_i}(y_i) \\ &= 2^{-n_i}(F(2^{\alpha n_i}(y_i - c_{n_i,k_i} + 2^{-n_i})) - F(2^{\alpha n_i}(y_i - c_{n_i,k_i}))) \\ &\leqslant M 2^{-n_i(2-\alpha)}. \end{aligned} \tag{1}$$

It follows that

$$\sum_i (\operatorname{diam} S_i)^{2-\alpha} = c \sum_i 2^{-n_i(2-\alpha)} \geqslant cM^{-1} \sum_i |P(S_i \cap \Gamma)| = cM^{-1},$$

where $c = 2^{(2-\alpha)/2}$, and hence $0 < \mathscr{H}^{2-\alpha}(\Gamma)$. The assertion follows from this and Lemma 2·1.

The crucial step of using the boundedness of F' is inequality (1). The following two lemmas are devised to show that under appropriate hypothesis, the squares that violate (1) are relatively few.

LEMMA 2·4. *For $0 < \delta < 1-\alpha$, let I be a dyadic interval in $[0,1)$ of size 2^{-n}, and let $\mathscr{S}_I \subseteq \mathscr{S}$ be the smallest disjoint collection of dyadic squares of size 2^{-n} that covers $P^{-1}(I) \cap \Gamma$. Let $\mathscr{S}'_I \subseteq \mathscr{S}_I$ consist of those $S \in \mathscr{S}_I$ such that*

$$|P(S \cap \Gamma)| > |I|^{2-\alpha-\delta}. \tag{2}$$

Then $\mathscr{S}'_I$ has at most $[2^{n(1-\alpha-\delta)}]$ members. ($[x]$ denotes the largest integer which is not greater than x.)

Remark. We refer to the above S the 'bad' squares. Since it needs $[c2^{n(1-\alpha)}]$ disjoint squares of size 2^{-n} to cover $P^{-1}(I) \cap \Gamma$, where c is some constant depending only on α, the portion of bad squares is at most $c2^{-n\delta}$.

Proof. Let q be the number of squares in $\mathscr{S}'_I$. Then

$$q|I|^{2-\alpha-\delta} < \sum_{S \in \mathscr{S}'_I} |P(S \cap \Gamma)| \leqslant |I|,$$

and hence $q \leqslant [2^{n(1-\alpha-\delta)}]$.

LEMMA 2·5. *Suppose that there exist $M > 0$ and $\epsilon_1 > 0$ such that for each $0 < \epsilon < \epsilon_1$, there exists $h(\epsilon) > 0$ with the property that*

$$\sum_{i=1}^{q} F(\alpha_i + h) - F(\alpha_i) \leqslant M q^{\epsilon_1} h^{\epsilon}, \tag{3}$$

for all $0 < h \leqslant h(\epsilon)$, and for all finite families of disjoint intervals $\{[\alpha_i, \alpha_i + h)\}_{i=1}^q$. Then $\dim \Gamma = 2-\alpha$.

Proof. Given $\delta > 0$, choose $0 < \epsilon < \epsilon_1$ close to ϵ_1 so that $\delta\epsilon_1 + (1-\alpha)(\epsilon - \epsilon_1) > 0$ and denote this number by η. Let $\bar{n} \in \mathbb{N}$ be such that

$$M \sum_{n=\bar{n}}^{\infty} 2^{-n\eta} < \tfrac{1}{3}. \tag{4}$$

Without loss of generality, we can assume that (3) holds for all $h \leqslant 2^{(\alpha-1)\bar{n}}$.

For any $n > \bar{n}$, let $I_{n,k}$ be a dyadic interval of $[0,1)$ and let $\mathscr{S}_{I_{n,k}}, \mathscr{S}'_{I_{n,k}}$ be defined as in Lemma 2·4. Let $B_{n,k} = \bigcup_i \{S_i : S_i \in \mathscr{S}'_{I_{n,k}}\}$ where $S_i = I_{n,k} \times [u_i, u_i + 2^{-n})$. Then

$$\begin{aligned}|P(B_{n,k} \cap \Gamma)| &= \sum_{S_i \in \mathscr{S}'_{I_{n,k}}} |P(S_i \cap \Gamma)| = \sum (F_{n,k}(u_i + 2^{-n}) - F_{n,k}(u_i)) \\ &= 2^{-n} \sum (F(2^{\alpha n}(u_i - c_{n,k} + 2^{-n})) - F(2^{\alpha n}(u_i - c_{n,k}))) \\ &\leqslant 2^{-n} M 2^{-n(\alpha+\delta-1)\epsilon_1} 2^{(\alpha-1)n\epsilon} = M 2^{-n(1+\eta)}.\end{aligned}$$

Here we have used Lemmas 2·2 and 2·4 and inequality (3). Define

$$B = \bigcup_{n \geqslant \bar{n}} \bigcup_{k=0}^{2^n-1} B_{n,k}.$$

Then the above estimate and the inequality (4) imply that

$$|P(B \cap \Gamma)| \leqslant M \sum_{n=\bar{n}}^{\infty} 2^{-n\eta} < \tfrac{1}{3}.$$

(This means that the projection of all the bad squares of size less than $2^{-\bar{n}}$ has Lebesgue measure less than $\frac{1}{3}$.)

Now let $\mathscr{C} \subseteq \mathscr{S}$ be an arbitrary cover of Γ such that each $S_i \in \mathscr{C}$ has size less than $2^{-\bar{n}}$. If $S_i \nsubseteq B$, then

$$|P(S_i)|^{2-\alpha-\delta} \geqslant |P(S_i \cap \Gamma)|,$$

and we have

$$\begin{aligned}\sum_{S_i \in \mathscr{C}} (\operatorname{diam} S_i)^{2-\alpha-\delta} &= c \sum_{S_i \in \mathscr{C}} |P(S_i)|^{2-\alpha-\delta} \geqslant c \sum_{S_i \in \mathscr{C},\, S_i \nsubseteq B} |P(S_i \cap \Gamma)| \\ &\geqslant c(1 - |P(B \cap \Gamma)|) \geqslant \tfrac{2}{3} c,\end{aligned}$$

where $c = 2^{(2-\alpha-\delta)/2}$. Hence $\mathscr{H}^{2-\alpha-\delta}(\Gamma) > 0$. This implies that $\dim \Gamma \geqslant 2 - \alpha - \delta$. Since δ is arbitrary, we conclude that $\dim \Gamma \geqslant 2 - \alpha$, and hence $\dim \Gamma = 2 - \alpha$ by Lemma 2·1.

THEOREM 2·6. *Suppose that there exists $p > 1$ such that for every $1 < \beta < p$,*

$$\lim_{h \to 0_+} \frac{1}{h^\beta} \int_{-\infty}^{\infty} |F(y+h) - F(y)|^p \, dy = 0. \tag{5}$$

Then $\dim \Gamma = 2 - \alpha$.

Proof. We first claim that for any $h > 0$,

$$F(a+h) - F(a) \leqslant \frac{1}{h} \int_{a-h}^{a+h} (F(x+h) - F(x)) \, dx.$$

Indeed,

$$\frac{1}{h}\int_{a-h}^{a+h}(F(x+h)-F(x))\,dx = \frac{1}{h}\left(\int_{a+h}^{a+2h}F(x)\,dx-\int_{a-h}^{a}F(x)\,dx\right)$$

$$\geqslant \frac{1}{h}\left(\left(xF(x)|_{a+h}^{a+2h}-\int_{a+h}^{a+2h}(a+2h)\,dF(x)\right)\right.$$

$$\left.-\left(xF(x)|_{a-h}^{a}-\int_{a-h}^{a}(a-h)\,dF(x)\right)\right)$$

$$= F(a+h)-F(a).$$

Now for any finite family of disjoint intervals $\{[a_i, a_i+h)\}_{i=1}^{q}$, with union U, writing $V = \bigcup_{i=1}^{q}[a_i-h, a_i)$ we have

$$\sum_{i=1}^{q}(F(a_i+h)-F(a_i)) \leqslant \frac{1}{h}\sum_{i=1}^{q}\int_{a_i-h}^{a_i+h}|F(x+h)-F(x)|\,dx$$

$$= \frac{1}{h}\left(\int_U+\int_V|F(x+h)-F(x)|\,dx\right)$$

$$\leqslant \frac{2}{h}(qh)^{1/p'}\left(\int_{-\infty}^{\infty}|F(x+h)-F(x)|^p\,dx\right)^{1/p}$$

$$= 2q^{1/p'}h^{(\beta-1)/p}\left(\frac{1}{h^{\beta}}\int_{-\infty}^{\infty}|F(x+h)-F(x)|^p\,dx\right)^{1/p},$$

where $1/p+1/p' = 1$, and $1 < \beta < p$. To check that the condition in Lemma 2·5 is fulfilled, we let $M = 1$, $\epsilon_1 = 1/p'$, $\epsilon = (\beta-1)/p$ for $1 < \beta < p$, and let $h(\epsilon)$ be such that for $0 < h \leqslant h(\epsilon)$,

$$2\left(\frac{1}{h^{\beta}}\int_{-\infty}^{\infty}|F(x+h)-F(x)|^p\,dx\right)^{1/p} \leqslant 1.$$

Proof of Theorem 1·1. The assertion is a corollary of Theorem 2·6 since F is absolutely continuous and $F' \in L^p$ for some $p > 1$ if and only if

$$\sup_{h\geqslant 0}\frac{1}{h^p}\int_{-\infty}^{\infty}|F(x+h)-F(x)|^p\,dx$$

is bounded (see [5]), hence (5) is satisfied. It can also be proved by a direct verification of (3). Let $E = \bigcup_{i=1}^{q}[a_i, a_i+h)$. Then

$$\sum_{i=1}^{q}(F(a_i+h)-F(a_i)) = \int_E F'(x)\,dx \leqslant |E|^{1/p'}\left(\int_{-\infty}^{\infty}|F'(x)|^p\,dx\right)^{1/p}$$

$$= \left(\int_{-\infty}^{\infty}|F'(x)|^p\,dx\right)^{1/p} q^{1/p'}h^{1/p'},$$

where $1/p+1/p' = 1$. By taking $\epsilon_1 = 1/p'$ in Lemma 2·5, condition (3) is clearly fulfilled.

3. *Some open questions*

The Fourier transformation of the distribution function F_α is given by

$$L_\alpha(u) = \prod_{i=1}^{\infty} \cos(2^{-\alpha i}u).$$

By studying the function $L_\alpha(u)$, Wintner[**10**] proved that if $\alpha = 1/n$, where $n \in \mathbb{N}$, then F_α is absolutely continuous. (Actually $F'_\alpha \in L^\infty$: see Garsia[**4**], theorem 1·8.) Erdös[**2**] proved that for any positive integer m there exists a $\beta(m) < 1$ (sufficiently close to 1) such that for almost all α with $\beta(m) < 2^{-\alpha} < 1$, F_α has an mth derivative. On the other hand by using some algebraic number theory, Salem[**9**] characterized the α such that $L_\alpha(u) \nrightarrow 0$ as $u \to \infty$ as those such that 2^α is a Pisot–Vijayaraghavan number. By the Riemann Lebesgue Lemma, such F_α cannot be absolutely continuous. Some more special cases including a necessary and sufficient condition for F_α to be absolutely continuous with a derivative in L^p for some $p > 1$ had also been obtained by Garsia[**4**]. Despite all these the following is still an open question.

Question 1. Is F_α absolutely continuous for almost all $\alpha \in (0, 1)$?

The condition in Theorem 2·6 is slightly weaker than the condition that F_α is absolutely continuous with $F'_\alpha \in L^p$ for some $p > 1$.

Question 2. Does F_α satisfy (5) for all $\alpha \in (0, 1)$?

Beyer[**1**] showed that the Hausdorff dimension of the level sets of f_α, with $\alpha = \frac{1}{2}, \frac{1}{3}, \ldots$, is $1-\alpha$ for almost all levels. Combining this result with an argument of Marstrand[3], theorem 5·8 it can be shown that $\dim \Gamma_{f_\alpha} = 2-\alpha$ for $\alpha = \frac{1}{2}, \frac{1}{3}, \ldots$.

Question 3. Does the absolute continuity of F_α imply that the level sets have dimension $1-\alpha$?

Question 4. Can the argument used in this note be applied to solve the dimension problem for the Weierstrass functions W_α or the Takagi functions

$$(T_\alpha(x) = \sum_{i=1}^{\infty} \lambda^{-\alpha i}\psi(\lambda^i x),$$

where $\lambda > 1$, $\psi(x)$ is of period 1 and equals $1-|1-2x|$ on $[0, 1]$)?

For sums of Rademacher functions other than the geometric sum, Beyer[**1**] proved that if the sequence $\{a_i\}$ is in l_2, but not in l_1, then $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes every preassigned real value on a set of Hausdorff dimension 1. This implies that its graph has Hausdorff dimension 2.

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